2.1 a: $m\ddot{x} + \gamma\dot{x} + kx = e^{i\omega t}$ forcing term often written as $F\cos(\omega t)$

For an undamped simple harmonic oscillator

$$F = ma = m\frac{d^2x}{dt^2} = m\ddot{x} = -kx$$

which is to say that force should be proportional to position (from 0) and pushing in the opposite direction of the displacement.

$\therefore$ the damping term must be $0 = \gamma$

2.1 b: the homogeneous case is:

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

using the ansatz $x(t) = e^{rt}$ we get always for linear systems

where r is a complex #

$$\frac{mr^2e^{rt} + \gamma re^{rt} + ke^{rt} = 0}{e^{rt}} \Rightarrow mr^2 + \gamma r + k = 0$$

where

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{}$$

$2m$

we can see that if

$$\gamma \geq 2\sqrt{mk}$$

we are overdamped: r is all real so no oscillations

$$\gamma = 2\sqrt{mk}$$

critically damped: we return to $x=0$ as fast as possible

$$\gamma \leq 2\sqrt{mk}$$

under damped: we will have oscillations with exponential growth or decay

 or

$\boxed{2.1\ c:}$ Now solving the general form (with non-zero forcing term)

$$m\ddot{x} + \gamma\dot{x} + kx = e^{i\omega t}$$

with ansatz $x(t) = Ae^{i\omega t}$

$$\cancel{e^{i\omega t}}\left[\ \ ^2 \ \ \ \ kA\right) = \cancel{e^{i\omega t}} = 1$$

$$e^{i\omega t}\left(mAi^2\omega^2 + \gamma A i\omega + kA\right) = e^{i\omega t}$$

$$mA\omega^2 + \gamma A i\omega + kA = 1$$

$$A = \frac{1}{-m\omega^2 + i\gamma\omega + k}$$

we could plug this in to solve for x directly, or in python we can plot frequency response to find resonance @ $\omega = 1$

2.1 d

2.2

undamped we know that

$$F = m\ddot{x}$$

Normal modes are both particles moving sinusoidally with same frequency

$\therefore$ for particles 1 & 2

$$F_1 = m\ddot{x}_1 = -kx_1 - k(x_1 - x_2)$$

$F_2 = m\ddot{x}_2 = -kx_2 - k(x_2 - x_1)$

$\therefore$

$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$

$m\ddot{x}_2 + 2kx_2 - kx_1 = 0$

$x = k/m$

In matrix form: divide by m: $\begin{bmatrix} 2x & -x \\ x & 2x \end{bmatrix}$

$$m\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

NUM $M^{-1} = \begin{bmatrix} 1/m & 0 \\ 0 & 1/m \end{bmatrix}$ or $M\ddot{x} = -Kx$

$$M^{-1}M\ddot{x} = M^{-1}Kx \Rightarrow \ddot{x} = M^{-1}KA$$

$|A - \lambda I| = 0$ characteristic equation of A to find eigenvalues

$$\begin{vmatrix} 2x & -x \\ -x & 2x \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \begin{vmatrix} 2x-\lambda & -x \\ -x & 2x-\lambda \end{vmatrix} = 0$$

determinant $ad - bc$

$$(2x - \lambda)(2x - \lambda) - x^2 = 0$$

$(4x^2 - 2x\lambda - 2x\lambda + \lambda^2 - x^2 = 0$

$$4x^2 - 2x\lambda - 2x\lambda + \lambda^2 - x^2 = 0$$

$$3x^2 - 4x\lambda + \lambda^2 = 0$$

$$\lambda = \frac{4x \pm \sqrt{16x^2 - 12x^2}}{2} = 2x \pm \frac{1}{2}\sqrt{4x^2}$$

$$\lambda = 2x \pm x$$

$\therefore$ normal modes

when $2\frac{k}{m} + \frac{k}{m} = 3\frac{k}{m}$

and

$$2\frac{k}{m} - \frac{k}{m} = \frac{k}{m}$$

$\boxed{2.3}$ discrete time

$$y(k) = \alpha \underbrace{y(k-1)}_{\text{prev } k} + (1-\alpha)\underbrace{x(k)}_{\text{current } k}$$

α is some constant where

$\alpha \approx 1$ means heavy filtering

& $\alpha \approx 0$ means light,

Z-transform is the discrete Laplace

Laplace reminder $\mathcal{L}\{f(t)\} \Rightarrow F(s) = \int_0^\infty e^{-st} f(t)\,dt$

easier to solve ODEs after laplace transform, then you move back

z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad z = re^{j\omega}$$

isolate y's

$$y(k) - \alpha y(k-1) = (1-\alpha) x(k)$$

identity

$$y(n-m) = \frac{z^{1-m}}{z-1}$$

NUM

$$\frac{z}{z-1} - \alpha \frac{z^{-1}}{z-1} = (1-\alpha) X(k)$$

$$\frac{z-\alpha}{z-1} = (1-\alpha) X(x) \Rightarrow X(k) =$$

$$\frac{z-\alpha}{(z-1)(1-\alpha)} = \frac{z-\alpha}{z-2\alpha-1+\alpha} = \frac{z-\alpha}{z-3\alpha-1}$$

[eq 4.53]

$$z(y(k)) = z(\alpha y(k-1)) + z((1-\alpha) x(k))$$

assumed to be 0

$$Z\{y(k)\} = \alpha\left(z^{-1} Y(z) + \cancel{y(-1)}\right) + (1-\alpha) X(z) = Y(z)$$

combine

$$z\, Y(z) = (1-\alpha) X(z) + \frac{\alpha}{z} Y(z)$$

Y(z) terms z

$$\frac{z-\alpha}{z} Y(z) = (1-\alpha) X(z)$$

$$Y(k) = (1-\alpha) \frac{z}{z-\alpha} X(k)$$

to use another identity, let's define $h(k) = \alpha^k$

$$\therefore Z\{\alpha^k\} = \frac{z}{z-\alpha}$$

$$\therefore Y(z) = (1-\alpha) H(z) X(z)$$

now inverse

eq. [4.55]

sub back in

$$y(k) = (1-\alpha) \sum_{n=0}^{k} x(n) h(k-n)$$

$$y(k) = (1-\alpha) \sum_{n=0}^{k} x(n) \alpha^{k-n}$$

for frequency response, $x(k) = e^{i\omega\delta_t k}$

from [eq 3.57]

Now back again

$$y(k) = e^{i\omega\delta_t k} H(e^{i\omega\delta_t})$$

$$y(k) = e^{i\omega\delta_t k} (1-\alpha) H(e^{i\omega\delta_t})$$

$$y(k) = e^{i\omega \delta_t k} (1-\alpha) \frac{e^{i\omega \delta_t}}{e^{i\omega \delta t} - \alpha}$$